Math 565: Functional Analysis
Lecture 22

Lemma. Let $\{u_i\}_{i\in I}$ be a orthonormal family in a Hilbert space H and let $M := \overline{\operatorname{span}\{u_i\}_{i\in I}}$. Then for each $x \in H$, $\operatorname{proj}_M x = \sum_{i\in I} \langle x, u_i\rangle u_i$; equivalently, $x - \sum_{i\in I} \langle x, u_i\rangle u_i \perp M$.

Proof. By Bessel's ineq, $\sum_{i\in I} |\langle x, u_i\rangle|^2 \le \|x\|^2$ so $(\langle x, u_i\rangle)_{i\in I} \in \ell^2(I)$, hence by HW, $\sum_{i\in I} \langle x, u_i\rangle u_i$ converges in H, hence it is $=$ some $y \in M$. For $x - y \perp M$, it is enough to check $x - y \perp u_i$ for each $i \in I$. But by the continuity of $\langle \cdot, \cdot \rangle$, we have:

$$\langle x-y, u_i\rangle = \langle x, u_i\rangle - \langle \sum_{j\in I} \langle x, u_j\rangle u_j, u_i\rangle = \langle x, u_i\rangle - \sum_{j\in I} \langle x, u_j\rangle \langle u_j, u_i\rangle = \langle x, u_i\rangle - \langle x, u_i\rangle = 0.$$

$\square$

Characterization of orthonormal bases. Let H be a Hilbert space and $\{u_i\}_{i\in I}$ be an ON family. Then TFAE:

(1') $\overline{\operatorname{span}\{u_i\}_{i\in I}} = H$.

(1) $\{u_i\}_{i\in I}$ is an ON basis.

(2) Parseval's identity: $\|x\|^2 = \sum_{i\in I} |\langle x, u_i\rangle|^2$ for each $x \in H$.

(3) For each $x \in H$, $(x \perp u_i$ for all $i \in I) \Rightarrow x = 0$.

Proof. (1') $\Rightarrow$ (1). $x = \operatorname{proj}_H x = \sum_{i\in I} \langle x, u_i\rangle u_i$ by the above lemma.

(1) $\Rightarrow$ (2). $\|x\|^2 = \langle \sum_{i\in I} \langle x, u_i\rangle u_i, \sum_{j\in I} \langle x, u_j\rangle u_j\rangle =$ (by the continuity of $\langle \cdot, \cdot \rangle$) $\sum_{i,j\in I} \langle x, u_i\rangle \overline{\langle x, u_j\rangle} \langle u_i, u_j\rangle$

$$= \sum_{i\in I} |\langle x, u_i\rangle|^2.$$

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1'). Let $M := \overline{\operatorname{span}\{u_i\}_{i\in I}}$. Then $M^\perp = \{u_i\}_{i\in I}^\perp = \{0\}$ and since $H = M + M^\perp$, we have $H = M$. $\square$

Theorem. Every Hilbert space admits an ON basis. Any two ON bases have the same cardinality.

Proof. For the existence, apply Zorn's lemma to get a maximal ON family $\{u_i\}_{i\in I}$ of nonzero vectors in H. Then maximality implies that if $x \perp \{u_i\}_{i\in I}$, then $x=0$, which is condition (3) above, so $\{u_i\}_{i\in I}$ is an ON basis.

Now let $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ be ON bases. It suffices to show $|I| \leq |J|$. We may assume I and J are infinite since for finite bases we already know this from the exchange property since then these are just linear (Hamel) bases.

For each v_j, $I_j := \{i \in I : \langle v_j, u_i\rangle \neq 0\}$ is ctbl so $I' := \bigcup_{j\in J} I_j$ has cardinality $\leq |J| \times |\mathbb{N}| \leq |J|\cdot|J| = |J|$ and $\{u_i\}_{i\in I'}$ is already an ON basis by (3) since if $x \perp \{u_i\}_{i\in I'}$ then $x \perp \{v_j\}_{j\in J}$, hence $x=0$. Thus, $I' = I$, so $|I| \leq |J|$. □

Def. For a Hilbert space H, define its Hilbert dimension (or orthonormal dimension) as the cardinality of some/any ON basis, denoted $\mathrm{Hdim}(H)$.

Thm. Every Hilbert space H is unitarily isomorphic (i.e. $\exists$ unitary map) to $\ell^2(\mathrm{Hdim}(H))$.

Proof. Fix an ON basis $\{e_i\}_{i\in I}$ and define $x \mapsto (\langle x, e_i\rangle)_{i\in I} : H \to \ell^2(I)$. By HWG Q1(a), this is a linear bijection, which is an isometry by Parseval's identity: $\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2$. By the polarization identity, this map preserves the inner product, hence is unitary. □

For a nonzero $x \in H$, denote $n(x) := \frac{1}{\|x\|}x$; for $x=0$, put $n(x) := 0$.

Gram-Schmidt process. Given a lin. indep. sequence $(x_n)_{n<N}$ in a Hilbert space H, there is a ON sequence $(u_n)_{n<N}$ such that $\forall k \leq N$, $\mathrm{span}(x_n)_{n<k} = \mathrm{span}(u_n)_{n<k}$. Namely, $u_0 := n(x_0)$ and $u_n := n(x_n - \sum_{i<n}\langle x_n, u_i\rangle u_i)$.

Proof. Supposing $(u_i)_{i<n}$ is defined, define u_n as above and recall that $\sum_{i<n}\langle x_n, u_i\rangle u_i = \mathrm{proj}_M x_n$ where $M = \mathrm{span}\{u_i\}_{i<n}$, so $x_n - \mathrm{proj}_M x_n \perp M$, so $u_n \perp \{u_i\}_{i<n}$. Furthermore $\mathrm{span}\{u_i\}_{i\leq n} = \mathrm{span}(\{u_n\} \cup \mathrm{span}\{u_i\}_{i<n}) = \mathrm{span}(\{x_n\} \cup \mathrm{span}\{u_i\}_{i<n})$ (by induction) $= \mathrm{span}(\{x_n\} \cup \mathrm{span}\{x_i\}_{i<n}) = \mathrm{span}\{x_i\}_{i\leq n}$. □

Cor. Every separable Hilbert space H admits a ctbl ON basis. Thus, every ∞-dim separable Hilbert space is unitarily isomorphic to $\ell^2(\mathbb{N})$.
In particular, $L^2(\mathbb{R}^d, \lambda) \cong \ell^2(\mathbb{N})$.

Proof. Let $D \subseteq H$ be ctbl dense set and let $\{x_n\}_{n<N} \subseteq D$ be a maximal linearly indep. subset, where $N \in \mathbb{N} \cup \{\infty\}$. Apply Gram-Schmidt to get an ON family $\{e_n\}_{n<N}$ with the same span so $\operatorname{span}\{e_n\}_{n<N} \supseteq D$ is dense, so by (1'), $\{e_n\}_{n<N}$ is an ON basis. $\square$

Fourier transform on the circle S^1

Let μ denote the Haar measure on $S^1 \subseteq \mathbb{C}$, the unit circle which is a group under complex multiplication. Also, $\mathbb{R}/\mathbb{Z} \cong S^1$ by $x \mapsto e^{2\pi i x} : \mathbb{R}/\mathbb{Z} \to S^1$. Hence μ is just the pushforward of Lebesgue measure on $[0,1)$ by $x \mapsto e^{2\pi i x}$.

Since $L^2(S^1, \mu)$ is separable (since (S^1, μ) is ctbly generated), we know abstractly that $L^2(S^1,\mu) \cong \ell^2(\mathbb{Z})$. However, there is a special isomorphism, which is called the Fourier transform.

Prop. Let $u_n \in L^2(S^1, \mu)$, $n \in \mathbb{Z}$, be the function $u_n(z) := z^n$, so $u_n(e^{2\pi i\theta}) = e^{2\pi i \theta n}$.
Then $\{u_n\}_{n\in\mathbb{N}}$ is an ON basis for $L^2(S^1,\mu)$.

Proof.
$$\langle z^n, z^m \rangle := \int_{S^1} z^n \bar{z}^m \, d\mu(z) = \int_{[0,1)} e^{2\pi i \theta n} e^{-2\pi i \theta m} \, d\lambda(\theta) = \int_0^1 e^{2\pi i \theta(n-m)} \, d\theta = \delta_{nm} := \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$$

Thus $\{u_n\}_{n\in\mathbb{N}}$ is ON. Remains to check that its span is dense. Even just $\{u_1\}$ as a function on S^1 separates points, so the algebra of all polynomials $\{z^n\}$ are dense in the uniform norm in $C(S^1)$, by the Stone-Weierstrass thm. Thus, it is dense in L^2 norm because μ is a finite measure. It remains to note that the algebra generated by the u_n is still just $\operatorname{span}\{u_n\}_{n\in\mathbb{N}}$ because $u_n \cdot u_m = u_{n+m}$. $\square$

For $f \in L^2(S^1,\mu)$, denote by $\hat{f} := (\langle f, u_n\rangle)_{n\in\mathbb{Z}}$, where $u_n(z) := z^n$. The vector $\hat{f} \in \ell^2(\mathbb{Z})$ is called the Fourier transform. Conversely, given $g \in \ell^2(\mathbb{Z})$, put $\check{g} := \sum_{n\in\mathbb{Z}} g(n) u_n$ and call it the inverse Fourier transform of g.

Cor. The map $f \mapsto \hat{f} : L^2(S^1,\mu) \to \ell^2(\mathbb{Z})$ is unitary.

Remark. Although $L^2(\mathbb{R},\lambda)$ is also separable, the Fourier transform on $L^2(\mathbb{R},\lambda)$ yields a function again in $L^2(\mathbb{R},\lambda)$ and not in $\ell^2(\mathbb{Z})$; in other words, Fourier transform has nothing to do with the Hilbert space structure in general and what happens on $L^2(S^1,\lambda)$ is somewhat a coincidence. Indeed, the functions $u_n(z) := z^n$ are exactly the group homomorphisms $S^1 \to S^1$. They form a group $\{u_n : n \in \mathbb{Z}\}$ under pointwise multiplication, which is isomorphic to $\mathbb{Z}$.
In general, for every locally compact abelian group G, its Pontryagin dual is the group $\hat{G}$ of all homomorphisms $\chi : G \to S^1$ under pointwise multiplication. This is again a locally compact abelian group wrt the pointwise convergence topology. Pontryagin duality theorem states that $\hat{\hat{G}}$ is canonically isomorphic to G. The Fourier transform is a linear isomorphism $f \mapsto \hat{f} : L^2(G,\mu_G) \to L^2(\hat{G},\mu_{\hat{G}})$, where μ_G and $\mu_{\hat{G}}$ are Haar measures on G and $\hat{G}$.
As we saw, the Pontryagin dual of S^1 is $\mathbb{Z}$ (in general, the Pontryagin dual of a compact group is discrete and vice versa), while the Pontryagin dual of $\mathbb{R}$ is again $\mathbb{R}$.